# Problem A.1

Consider the ordinary vectors in three dimensions $\left(a_x\hat{\imath} + a_y\hat{\jmath} + a_z\hat{k}\right)$, with complex components.

**(a)** Does the subset of all vectors with $a_z = 0$ constitute a vector space? If so, what is its dimension; if not, why not?

**(b)** What about the subset of all vectors whose $z$ component is 1? *Hint:* Would the sum of two such vectors be in the subset? How about the null vector?

**(c)** What about the subset of vectors whose components are all equal?

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**Solution**

In order for a collection of vectors $\mathcal{V}$ to be a vector space over the complex numbers $\mathbb{C}$, the vector addition and scalar multiplication operations defined on it must satisfy the following ten properties.

(A1) $\mathbf{x} + \mathbf{y} \in \mathcal{V}$ for all $\mathbf{x}, \mathbf{y} \in \mathcal{V}$.

(A2) $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$ for every $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathcal{V}$.

(A3) $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$ for every $\mathbf{x}, \mathbf{y} \in \mathcal{V}$.

(A4) There is an element $\mathbf{0} \in \mathcal{V}$ such that $\mathbf{x} + \mathbf{0} = \mathbf{x}$ for every $\mathbf{x} \in \mathcal{V}$.

(A5) For each $\mathbf{x} \in \mathcal{V}$, there is an element $(-\mathbf{x}) \in \mathcal{V}$ such that $\mathbf{x} + (-\mathbf{x}) = \mathbf{0}$.

(M1) $\alpha\mathbf{x} \in \mathcal{V}$ for all $\alpha \in \mathbb{C}$ and $\mathbf{x} \in \mathcal{V}$.

(M2) $(\alpha\beta)\mathbf{x} = \alpha(\beta\mathbf{x})$ for all $\alpha, \beta \in \mathbb{C}$ and every $\mathbf{x} \in \mathcal{V}$.

(M3) $\alpha(\mathbf{x} + \mathbf{y}) = \alpha\mathbf{x} + \alpha\mathbf{y}$ for every $\alpha \in \mathbb{C}$ and all $\mathbf{x}, \mathbf{y} \in \mathcal{V}$.

(M4) $(\alpha + \beta)\mathbf{x} = \alpha\mathbf{x} + \beta\mathbf{x}$ for all $\alpha, \beta \in \mathbb{C}$ and every $\mathbf{x} \in \mathcal{V}$.

(M5) $1\mathbf{x} = \mathbf{x}$ for every $\mathbf{x} \in \mathcal{V}$.

## Part (a)

Here $\mathcal{V}$ consists of all the vectors with three components that have a zero third component. Let $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ be vectors in $\mathcal{V}$ and let $\alpha$ and $\beta$ be complex scalars.

$$\mathbf{x} = x_1\hat{\mathbf{x}} + y_1\hat{\mathbf{y}} + 0\hat{\mathbf{z}}$$

$$\mathbf{y} = x_2\hat{\mathbf{x}} + y_2\hat{\mathbf{y}} + 0\hat{\mathbf{z}}$$

$$\mathbf{z} = x_3\hat{\mathbf{x}} + y_3\hat{\mathbf{y}} + 0\hat{\mathbf{z}}$$

Property A1

Take the sum of $\mathbf{x}$ and $\mathbf{y}$.

$$\mathbf{x} + \mathbf{y} = (x_1\hat{\mathbf{x}} + y_1\hat{\mathbf{y}} + 0\hat{\mathbf{z}}) + (x_2\hat{\mathbf{x}} + y_2\hat{\mathbf{y}} + 0\hat{\mathbf{z}}) = (x_1 + x_2)\hat{\mathbf{x}} + (y_1 + y_2)\hat{\mathbf{y}} + 0\hat{\mathbf{z}}$$

Because the $z$-component of $\mathbf{x} + \mathbf{y}$ is also zero, $\mathbf{x} + \mathbf{y}$ is a vector in $\mathcal{V}$. Property A1 is satisfied.

Property A2

Compare the vector sums of $(\mathbf{x}+\mathbf{y})+\mathbf{z}$ and $\mathbf{x}+(\mathbf{y}+\mathbf{z})$.

$$\begin{aligned}
(\mathbf{x}+\mathbf{y})+\mathbf{z} &= [(x_1\hat{\mathbf{x}}+y_1\hat{\mathbf{y}}+0\hat{\mathbf{z}})+(x_2\hat{\mathbf{x}}+y_2\hat{\mathbf{y}}+0\hat{\mathbf{z}})]+(x_3\hat{\mathbf{x}}+y_3\hat{\mathbf{y}}+0\hat{\mathbf{z}}) \\
&= [(x_1+x_2)\hat{\mathbf{x}}+(y_1+y_2)\hat{\mathbf{y}}+0\hat{\mathbf{z}}]+(x_3\hat{\mathbf{x}}+y_3\hat{\mathbf{y}}+0\hat{\mathbf{z}}) \\
&= [(x_1+x_2)+x_3]\hat{\mathbf{x}}+[(y_1+y_2)+y_3]\hat{\mathbf{y}}+0\hat{\mathbf{z}} \\
&= (x_1+x_2+x_3)\hat{\mathbf{x}}+(y_1+y_2+y_3)\hat{\mathbf{y}}+0\hat{\mathbf{z}} \\
\mathbf{x}+(\mathbf{y}+\mathbf{z}) &= (x_1\hat{\mathbf{x}}+y_1\hat{\mathbf{y}}+0\hat{\mathbf{z}})+[(x_2\hat{\mathbf{x}}+y_2\hat{\mathbf{y}}+0\hat{\mathbf{z}})+(x_3\hat{\mathbf{x}}+y_3\hat{\mathbf{y}}+0\hat{\mathbf{z}})] \\
&= (x_1\hat{\mathbf{x}}+y_1\hat{\mathbf{y}}+0\hat{\mathbf{z}})+[(x_2+x_3)\hat{\mathbf{x}}+(y_2+y_3)\hat{\mathbf{y}}+0\hat{\mathbf{z}}] \\
&= [x_1+(x_2+x_3)]\hat{\mathbf{x}}+[y_1+(y_2+y_3)]\hat{\mathbf{y}}+0\hat{\mathbf{z}} \\
&= (x_1+x_2+x_3)\hat{\mathbf{x}}+(y_1+y_2+y_3)\hat{\mathbf{y}}+0\hat{\mathbf{z}}
\end{aligned}$$

Because of the associative property of addition, $(\mathbf{x}+\mathbf{y})+\mathbf{z}=\mathbf{x}+(\mathbf{y}+\mathbf{z})$. Property A2 is satisfied.

Property A3

Compare the vector sums of $\mathbf{x}+\mathbf{y}$ and $\mathbf{y}+\mathbf{x}$.

$$\begin{aligned}
\mathbf{x}+\mathbf{y} &= (x_1\hat{\mathbf{x}}+y_1\hat{\mathbf{y}}+0\hat{\mathbf{z}})+(x_2\hat{\mathbf{x}}+y_2\hat{\mathbf{y}}+0\hat{\mathbf{z}}) \\
&= (x_1+x_2)\hat{\mathbf{x}}+(y_1+y_2)\hat{\mathbf{y}}+0\hat{\mathbf{z}} \\
\mathbf{y}+\mathbf{x} &= (x_2\hat{\mathbf{x}}+y_2\hat{\mathbf{y}}+0\hat{\mathbf{z}})+(x_1\hat{\mathbf{x}}+y_1\hat{\mathbf{y}}+0\hat{\mathbf{z}}) \\
&= (x_2+x_1)\hat{\mathbf{x}}+(y_2+y_1)\hat{\mathbf{y}}+0\hat{\mathbf{z}} \\
&= (x_1+x_2)\hat{\mathbf{x}}+(y_1+y_2)\hat{\mathbf{y}}+0\hat{\mathbf{z}}
\end{aligned}$$

Because of the commutative property of addition, $\mathbf{x}+\mathbf{y}=\mathbf{y}+\mathbf{x}$. Property A3 is satisfied.

Property A4

The zero element is the vector in $\mathcal{V}$ with zero $x$- and $y$-components.

$$\mathbf{0}=0\hat{\mathbf{x}}+0\hat{\mathbf{y}}+0\hat{\mathbf{z}}$$

Adding this to $\mathbf{x}$ results in $\mathbf{x}$.

$$\begin{aligned}
\mathbf{x}+\mathbf{0} &= (x_1\hat{\mathbf{x}}+y_1\hat{\mathbf{y}}+0\hat{\mathbf{z}})+(0\hat{\mathbf{x}}+0\hat{\mathbf{y}}+0\hat{\mathbf{z}}) \\
&= (x_1+0)\hat{\mathbf{x}}+(y_1+0)\hat{\mathbf{y}}+(0+0)\hat{\mathbf{z}} \\
&= x_1\hat{\mathbf{x}}+y_1\hat{\mathbf{y}}+0\hat{\mathbf{z}} \\
&= \mathbf{x}
\end{aligned}$$

Property A4 is satisfied.

Property A5

The additive inverse of $\mathbf{x}$ is $-\mathbf{x} = -(x_1\hat{\mathbf{x}} + y_1\hat{\mathbf{y}} + 0\hat{\mathbf{z}}) = -x_1\hat{\mathbf{x}} - y_1\hat{\mathbf{y}} + 0\hat{\mathbf{z}}$. Because the $z$-component of $-\mathbf{x}$ is zero, $-\mathbf{x}$ is also in $\mathcal{V}$.

$$\begin{aligned}
\mathbf{x} + (-\mathbf{x}) &= (x_1\hat{\mathbf{x}} + y_1\hat{\mathbf{y}} + 0\hat{\mathbf{z}}) + (-x_1\hat{\mathbf{x}} - y_1\hat{\mathbf{y}} + 0\hat{\mathbf{z}}) \\
&= [x_1 + (-x_1)]\hat{\mathbf{x}} + [y_1 + (-y_1)]\hat{\mathbf{y}} + 0\hat{\mathbf{z}} \\
&= 0\hat{\mathbf{x}} + 0\hat{\mathbf{y}} + 0\hat{\mathbf{z}} \\
&= \mathbf{0}
\end{aligned}$$

Property A5 is satisfied.

Property M1

Multiply $\alpha$ and $\mathbf{x}$.

$$\begin{aligned}
\alpha\mathbf{x} &= \alpha(x_1\hat{\mathbf{x}} + y_1\hat{\mathbf{y}} + 0\hat{\mathbf{z}}) \\
&= \alpha x_1\hat{\mathbf{x}} + \alpha y_1\hat{\mathbf{y}} + 0\hat{\mathbf{z}}
\end{aligned}$$

Because the $z$-component of $\alpha\mathbf{x}$ is zero, $\alpha\mathbf{x}$ is in $\mathcal{V}$. Property M1 is satisfied.

Property M2

Compare the formulas of $(\alpha\beta)\mathbf{x}$ and $\alpha(\beta\mathbf{x})$.

$$\begin{aligned}
(\alpha\beta)\mathbf{x} &= (\alpha\beta)(x_1\hat{\mathbf{x}} + y_1\hat{\mathbf{y}} + 0\hat{\mathbf{z}}) \\
&= [(\alpha\beta)x_1]\hat{\mathbf{x}} + [(\alpha\beta)y_1]\hat{\mathbf{y}} + 0\hat{\mathbf{z}} \\
&= \alpha\beta x_1\hat{\mathbf{x}} + \alpha\beta y_1\hat{\mathbf{y}} + 0\hat{\mathbf{z}} \\
\alpha(\beta\mathbf{x}) &= \alpha[\beta(x_1\hat{\mathbf{x}} + y_1\hat{\mathbf{y}} + 0\hat{\mathbf{z}})] \\
&= \alpha(\beta x_1\hat{\mathbf{x}} + \beta y_1\hat{\mathbf{y}} + 0\hat{\mathbf{z}}) \\
&= [\alpha(\beta x_1)]\hat{\mathbf{x}} + [\alpha(\beta y_1)]\hat{\mathbf{y}} + 0\hat{\mathbf{z}} \\
&= \alpha\beta x_1\hat{\mathbf{x}} + \alpha\beta y_1\hat{\mathbf{y}} + 0\hat{\mathbf{z}}
\end{aligned}$$

Because of the associative property of multiplication, $(\alpha\beta)\mathbf{x} = \alpha(\beta\mathbf{x})$. Property M2 is satisfied.

Property M3

Compare the formulas of $\alpha(\mathbf{x} + \mathbf{y})$ and $\alpha\mathbf{x} + \alpha\mathbf{y}$.

$$\begin{aligned}
\alpha(\mathbf{x} + \mathbf{y}) &= \alpha[(x_1\hat{\mathbf{x}} + y_1\hat{\mathbf{y}} + 0\hat{\mathbf{z}}) + (x_2\hat{\mathbf{x}} + y_2\hat{\mathbf{y}} + 0\hat{\mathbf{z}})] \\
&= \alpha[(x_1 + x_2)\hat{\mathbf{x}} + (y_1 + y_2)\hat{\mathbf{y}} + 0\hat{\mathbf{z}}] \\
&= \alpha(x_1 + x_2)\hat{\mathbf{x}} + \alpha(y_1 + y_2)\hat{\mathbf{y}} + 0\hat{\mathbf{z}}
\end{aligned}$$

Evaluate the formula for $\alpha\mathbf{x} + \alpha\mathbf{y}$.

$$\begin{aligned}
\alpha\mathbf{x} + \alpha\mathbf{y} &= \alpha\left(x_1\hat{\mathbf{x}} + y_1\hat{\mathbf{y}} + 0\hat{\mathbf{z}}\right) + \alpha\left(x_2\hat{\mathbf{x}} + y_2\hat{\mathbf{y}} + 0\hat{\mathbf{z}}\right) \\
&= \left(\alpha x_1\hat{\mathbf{x}} + \alpha y_1\hat{\mathbf{y}} + 0\hat{\mathbf{z}}\right) + \left(\alpha x_2\hat{\mathbf{x}} + \alpha y_2\hat{\mathbf{y}} + 0\hat{\mathbf{z}}\right) \\
&= (\alpha x_1 + \alpha x_2)\hat{\mathbf{x}} + (\alpha y_1 + \alpha y_2)\hat{\mathbf{y}} + 0\hat{\mathbf{z}}
\end{aligned}$$

Because of the distributive property, $\alpha(\mathbf{x}+\mathbf{y}) = \alpha\mathbf{x} + \alpha\mathbf{y}$. Property M3 is satisfied.

Property M4

Compare the formulas of $(\alpha+\beta)\mathbf{x}$ and $\alpha\mathbf{x} + \beta\mathbf{x}$.

$$\begin{aligned}
(\alpha+\beta)\mathbf{x} &= (\alpha+\beta)\left(x_1\hat{\mathbf{x}} + y_1\hat{\mathbf{y}} + 0\hat{\mathbf{z}}\right) \\
&= [(\alpha+\beta)x_1]\hat{\mathbf{x}} + [(\alpha+\beta)y_1]\hat{\mathbf{y}} + 0\hat{\mathbf{z}} \\
&= (\alpha x_1 + \beta x_1)\hat{\mathbf{x}} + (\alpha y_1 + \beta y_1)\hat{\mathbf{y}} + 0\hat{\mathbf{z}} \\
\alpha\mathbf{x} + \beta\mathbf{x} &= \alpha\left(x_1\hat{\mathbf{x}} + y_1\hat{\mathbf{y}} + 0\hat{\mathbf{z}}\right) + \beta\left(x_1\hat{\mathbf{x}} + y_1\hat{\mathbf{y}} + 0\hat{\mathbf{z}}\right) \\
&= \left(\alpha x_1\hat{\mathbf{x}} + \alpha y_1\hat{\mathbf{y}} + 0\hat{\mathbf{z}}\right) + \left(\beta x_1\hat{\mathbf{x}} + \beta y_1\hat{\mathbf{y}} + 0\hat{\mathbf{z}}\right) \\
&= (\alpha x_1 + \beta x_1)\hat{\mathbf{x}} + (\alpha y_1 + \beta y_1)\hat{\mathbf{y}} + 0\hat{\mathbf{z}}
\end{aligned}$$

Because of the distributive property, $(\alpha+\beta)\mathbf{x} = \alpha\mathbf{x} + \beta\mathbf{x}$. Property M4 is satisfied.

Property M5

Multiply 1 and $\mathbf{x}$.

$$\begin{aligned}
1\mathbf{x} &= 1\left(x_1\hat{\mathbf{x}} + y_1\hat{\mathbf{y}} + 0\hat{\mathbf{z}}\right) \\
&= x_1\hat{\mathbf{x}} + y_1\hat{\mathbf{y}} + 0\hat{\mathbf{z}} \\
&= \mathbf{x}
\end{aligned}$$

Property M5 is satisfied. All ten properties are satisfied, so the set of vectors with three components, the third being zero, is a vector space over the complex numbers. The dimension of the vector space is 2 because each of the vectors is spanned by the two basis vectors, $\hat{\mathbf{x}} = (1,0,0)$ and $\hat{\mathbf{y}} = (0,1,0)$.

## Part (b)

Here $\mathcal{V}$ consists of all the vectors with three components that have a third component of 1. Let $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ be vectors in $\mathcal{V}$ and let $\alpha$ and $\beta$ be complex scalars.

$$\begin{aligned}
\mathbf{x} &= x_1\hat{\mathbf{x}} + y_1\hat{\mathbf{y}} + 1\hat{\mathbf{z}} \\
\mathbf{y} &= x_2\hat{\mathbf{x}} + y_2\hat{\mathbf{y}} + 1\hat{\mathbf{z}} \\
\mathbf{z} &= x_3\hat{\mathbf{x}} + y_3\hat{\mathbf{y}} + 1\hat{\mathbf{z}}
\end{aligned}$$

Property A1

Take the sum of $\mathbf{x}$ and $\mathbf{y}$.

$$\mathbf{x}+\mathbf{y}=(x_1\hat{\mathbf{x}}+y_1\hat{\mathbf{y}}+1\hat{\mathbf{z}})+(x_2\hat{\mathbf{x}}+y_2\hat{\mathbf{y}}+1\hat{\mathbf{z}})=(x_1+x_2)\hat{\mathbf{x}}+(y_1+y_2)\hat{\mathbf{y}}+2\hat{\mathbf{z}}$$

Because the $z$-component of $\mathbf{x}+\mathbf{y}$ is not also 1, $\mathbf{x}+\mathbf{y}$ is not a vector in $\mathcal{V}$. Property A1 is not satisfied, so $\mathcal{V}$ is not a vector space over the complex numbers. Property A4 is also not satisfied because no choice of the first two components gives the zero element $\mathbf{0}=0\hat{\mathbf{x}}+0\hat{\mathbf{y}}+0\hat{\mathbf{z}}$.

## Part (c)

Here $\mathcal{V}$ consists of all the vectors with three equal components. Let $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ be vectors in $\mathcal{V}$ and let $\alpha$ and $\beta$ be complex scalars.

$$\mathbf{x}=a_1\hat{\mathbf{x}}+a_1\hat{\mathbf{y}}+a_1\hat{\mathbf{z}}$$

$$\mathbf{y}=a_2\hat{\mathbf{x}}+a_2\hat{\mathbf{y}}+a_2\hat{\mathbf{z}}$$

$$\mathbf{z}=a_3\hat{\mathbf{x}}+a_3\hat{\mathbf{y}}+a_3\hat{\mathbf{z}}$$

Property A1

Take the sum of $\mathbf{x}$ and $\mathbf{y}$.

$$\mathbf{x}+\mathbf{y}=(a_1\hat{\mathbf{x}}+a_1\hat{\mathbf{y}}+a_1\hat{\mathbf{z}})+(a_2\hat{\mathbf{x}}+a_2\hat{\mathbf{y}}+a_2\hat{\mathbf{z}})=(a_1+a_2)\hat{\mathbf{x}}+(a_1+a_2)\hat{\mathbf{y}}+(a_1+a_2)\hat{\mathbf{z}}$$

Because the components of $\mathbf{x}+\mathbf{y}$ are all equal, $\mathbf{x}+\mathbf{y}$ is a vector in $\mathcal{V}$. Property A1 is satisfied.

Property A2

Compare the vector sums of $(\mathbf{x}+\mathbf{y})+\mathbf{z}$ and $\mathbf{x}+(\mathbf{y}+\mathbf{z})$.

$$\begin{aligned}
(\mathbf{x}+\mathbf{y})+\mathbf{z}&=[(a_1\hat{\mathbf{x}}+a_1\hat{\mathbf{y}}+a_1\hat{\mathbf{z}})+(a_2\hat{\mathbf{x}}+a_2\hat{\mathbf{y}}+a_2\hat{\mathbf{z}})]+(a_3\hat{\mathbf{x}}+a_3\hat{\mathbf{y}}+a_3\hat{\mathbf{z}})\\
&=[(a_1+a_2)\hat{\mathbf{x}}+(a_1+a_2)\hat{\mathbf{y}}+(a_1+a_2)\hat{\mathbf{z}}]+(a_3\hat{\mathbf{x}}+a_3\hat{\mathbf{y}}+a_3\hat{\mathbf{z}})\\
&=[(a_1+a_2)+a_3]\hat{\mathbf{x}}+[(a_1+a_2)+a_3]\hat{\mathbf{y}}+[(a_1+a_2)+a_3]\hat{\mathbf{z}}\\
&=(a_1+a_2+a_3)\hat{\mathbf{x}}+(a_1+a_2+a_3)\hat{\mathbf{y}}+(a_1+a_2+a_3)\hat{\mathbf{z}}\\
\mathbf{x}+(\mathbf{y}+\mathbf{z})&=(a_1\hat{\mathbf{x}}+a_1\hat{\mathbf{y}}+a_1\hat{\mathbf{z}})+[(a_2\hat{\mathbf{x}}+a_2\hat{\mathbf{y}}+a_2\hat{\mathbf{z}})+(a_3\hat{\mathbf{x}}+a_3\hat{\mathbf{y}}+a_3\hat{\mathbf{z}})]\\
&=(a_1\hat{\mathbf{x}}+a_1\hat{\mathbf{y}}+a_1\hat{\mathbf{z}})+[(a_2+a_3)\hat{\mathbf{x}}+(a_2+a_3)\hat{\mathbf{y}}+(a_2+a_3)\hat{\mathbf{z}}]\\
&=[a_1+(a_2+a_3)]\hat{\mathbf{x}}+[a_1+(a_2+a_3)]\hat{\mathbf{y}}+[a_1+(a_2+a_3)]\hat{\mathbf{z}}\\
&=(a_1+a_2+a_3)\hat{\mathbf{x}}+(a_1+a_2+a_3)\hat{\mathbf{y}}+(a_1+a_2+a_3)\hat{\mathbf{z}}
\end{aligned}$$

Because of the associative property of addition, $(\mathbf{x}+\mathbf{y})+\mathbf{z}=\mathbf{x}+(\mathbf{y}+\mathbf{z})$. Property A2 is satisfied.

Property A3

Compare the vector sums of $\mathbf{x}+\mathbf{y}$ and $\mathbf{y}+\mathbf{x}$.

$$\begin{aligned}
\mathbf{x}+\mathbf{y} &= (a_1\hat{\mathbf{x}} + a_1\hat{\mathbf{y}} + a_1\hat{\mathbf{z}}) + (a_2\hat{\mathbf{x}} + a_2\hat{\mathbf{y}} + a_2\hat{\mathbf{z}}) \\
&= (a_1 + a_2)\hat{\mathbf{x}} + (a_1 + a_2)\hat{\mathbf{y}} + (a_1 + a_2)\hat{\mathbf{z}} \\
\mathbf{y}+\mathbf{x} &= (a_2\hat{\mathbf{x}} + a_2\hat{\mathbf{y}} + a_2\hat{\mathbf{z}}) + (a_1\hat{\mathbf{x}} + a_1\hat{\mathbf{y}} + a_1\hat{\mathbf{z}}) \\
&= (a_2 + a_1)\hat{\mathbf{x}} + (a_2 + a_1)\hat{\mathbf{y}} + (a_2 + a_1)\hat{\mathbf{z}} \\
&= (a_1 + a_2)\hat{\mathbf{x}} + (a_1 + a_2)\hat{\mathbf{y}} + (a_1 + a_2)\hat{\mathbf{z}}
\end{aligned}$$

Because of the commutative property of addition, $\mathbf{x}+\mathbf{y} = \mathbf{y}+\mathbf{x}$. Property A3 is satisfied.

Property A4

The zero element is the vector in $\mathcal{V}$ that has all components equal to zero.

$$\mathbf{0} = 0\hat{\mathbf{x}} + 0\hat{\mathbf{y}} + 0\hat{\mathbf{z}}$$

Adding this to $\mathbf{x}$ results in $\mathbf{x}$.

$$\begin{aligned}
\mathbf{x}+\mathbf{0} &= (a_1\hat{\mathbf{x}} + a_1\hat{\mathbf{y}} + a_1\hat{\mathbf{z}}) + (0\hat{\mathbf{x}} + 0\hat{\mathbf{y}} + 0\hat{\mathbf{z}}) \\
&= (a_1 + 0)\hat{\mathbf{x}} + (a_1 + 0)\hat{\mathbf{y}} + (a_1 + 0)\hat{\mathbf{z}} \\
&= a_1\hat{\mathbf{x}} + a_1\hat{\mathbf{y}} + a_1\hat{\mathbf{z}} \\
&= \mathbf{x}
\end{aligned}$$

Property A4 is satisfied.

Property A5

The additive inverse of $\mathbf{x}$ is $-\mathbf{x} = -\left(a_1\hat{\mathbf{x}} + a_1\hat{\mathbf{y}} + a_1\hat{\mathbf{z}}\right) = -a_1\hat{\mathbf{x}} - a_1\hat{\mathbf{y}} - a_1\hat{\mathbf{z}}$. Because the components of $-\mathbf{x}$ are equal, $-\mathbf{x}$ is also in $\mathcal{V}$.

$$\begin{aligned}
\mathbf{x}+(-\mathbf{x}) &= (a_1\hat{\mathbf{x}} + a_1\hat{\mathbf{y}} + a_1\hat{\mathbf{z}}) + (-a_1\hat{\mathbf{x}} - a_1\hat{\mathbf{y}} - a_1\hat{\mathbf{z}}) \\
&= [a_1 + (-a_1)]\hat{\mathbf{x}} + [a_1 + (-a_1)]\hat{\mathbf{y}} + [a_1 + (-a_1)]\hat{\mathbf{z}} \\
&= 0\hat{\mathbf{x}} + 0\hat{\mathbf{y}} + 0\hat{\mathbf{z}} \\
&= \mathbf{0}
\end{aligned}$$

Property A5 is satisfied.

Property M1

Multiply $\alpha$ and $\mathbf{x}$.

$$\begin{aligned}
\alpha\mathbf{x} &= \alpha\left(a_1\hat{\mathbf{x}} + a_1\hat{\mathbf{y}} + a_1\hat{\mathbf{z}}\right) \\
&= \alpha a_1\hat{\mathbf{x}} + \alpha a_1\hat{\mathbf{y}} + \alpha a_1\hat{\mathbf{z}}
\end{aligned}$$

Because the components of $\alpha\mathbf{x}$ are all equal, $\alpha\mathbf{x}$ is in $\mathcal{V}$. Property M1 is satisfied.

Property M2

Compare the formulas of $(\alpha\beta)\mathbf{x}$ and $\alpha(\beta\mathbf{x})$.

$$\begin{aligned}
(\alpha\beta)\mathbf{x} &= (\alpha\beta)\left(a_1\hat{\mathbf{x}} + a_1\hat{\mathbf{y}} + a_1\hat{\mathbf{z}}\right) \\
&= [(\alpha\beta)a_1]\hat{\mathbf{x}} + [(\alpha\beta)a_1]\hat{\mathbf{y}} + [(\alpha\beta)a_1]\hat{\mathbf{z}} \\
&= \alpha\beta a_1\hat{\mathbf{x}} + \alpha\beta a_1\hat{\mathbf{y}} + \alpha\beta a_1\hat{\mathbf{z}} \\
\alpha(\beta\mathbf{x}) &= \alpha\left[\beta\left(a_1\hat{\mathbf{x}} + a_1\hat{\mathbf{y}} + a_1\hat{\mathbf{z}}\right)\right] \\
&= \alpha\left(\beta a_1\hat{\mathbf{x}} + \beta a_1\hat{\mathbf{y}} + \beta a_1\hat{\mathbf{z}}\right) \\
&= [\alpha(\beta a_1)]\hat{\mathbf{x}} + [\alpha(\beta a_1)]\hat{\mathbf{y}} + [\alpha(\beta a_1)]\hat{\mathbf{z}} \\
&= \alpha\beta a_1\hat{\mathbf{x}} + \alpha\beta a_1\hat{\mathbf{y}} + \alpha\beta a_1\hat{\mathbf{z}}
\end{aligned}$$

Because of the associative property of multiplication, $(\alpha\beta)\mathbf{x} = \alpha(\beta\mathbf{x})$. Property M2 is satisfied.

Property M3

Compare the formulas of $\alpha(\mathbf{x}+\mathbf{y})$ and $\alpha\mathbf{x}+\alpha\mathbf{y}$.

$$\begin{aligned}
\alpha(\mathbf{x}+\mathbf{y}) &= \alpha\left[\left(a_1\hat{\mathbf{x}} + a_1\hat{\mathbf{y}} + a_1\hat{\mathbf{z}}\right) + \left(a_2\hat{\mathbf{x}} + a_2\hat{\mathbf{y}} + a_2\hat{\mathbf{z}}\right)\right] \\
&= \alpha\left[(a_1+a_2)\hat{\mathbf{x}} + (a_1+a_2)\hat{\mathbf{y}} + (a_1+a_2)\hat{\mathbf{z}}\right] \\
&= \alpha(a_1+a_2)\hat{\mathbf{x}} + \alpha(a_1+a_2)\hat{\mathbf{y}} + \alpha(a_1+a_2)\hat{\mathbf{z}}
\end{aligned}$$

Evaluate the formula for $\alpha\mathbf{x}+\alpha\mathbf{y}$.

$$\begin{aligned}
\alpha\mathbf{x}+\alpha\mathbf{y} &= \alpha\left(a_1\hat{\mathbf{x}} + a_1\hat{\mathbf{y}} + a_1\hat{\mathbf{z}}\right) + \alpha\left(a_2\hat{\mathbf{x}} + a_2\hat{\mathbf{y}} + a_2\hat{\mathbf{z}}\right) \\
&= \left(\alpha a_1\hat{\mathbf{x}} + \alpha a_1\hat{\mathbf{y}} + \alpha a_1\hat{\mathbf{z}}\right) + \left(\alpha a_2\hat{\mathbf{x}} + \alpha a_2\hat{\mathbf{y}} + \alpha a_2\hat{\mathbf{z}}\right) \\
&= (\alpha a_1 + \alpha a_2)\hat{\mathbf{x}} + (\alpha a_1 + \alpha a_2)\hat{\mathbf{y}} + (\alpha a_1 + \alpha a_2)\hat{\mathbf{z}}
\end{aligned}$$

Because of the distributive property, $\alpha(\mathbf{x}+\mathbf{y}) = \alpha\mathbf{x}+\alpha\mathbf{y}$. Property M3 is satisfied.

Property M4

Compare the formulas of $(\alpha+\beta)\mathbf{x}$ and $\alpha\mathbf{x}+\beta\mathbf{x}$.

$$\begin{aligned}
(\alpha+\beta)\mathbf{x} &= (\alpha+\beta)\left(a_1\hat{\mathbf{x}} + a_1\hat{\mathbf{y}} + a_1\hat{\mathbf{z}}\right) \\
&= [(\alpha+\beta)a_1]\hat{\mathbf{x}} + [(\alpha+\beta)a_1]\hat{\mathbf{y}} + [(\alpha+\beta)a_1]\hat{\mathbf{z}} \\
&= (\alpha a_1 + \beta a_1)\hat{\mathbf{x}} + (\alpha a_1 + \beta a_1)\hat{\mathbf{y}} + (\alpha a_1 + \beta a_1)\hat{\mathbf{z}} \\
\alpha\mathbf{x}+\beta\mathbf{x} &= \alpha\left(a_1\hat{\mathbf{x}} + a_1\hat{\mathbf{y}} + a_1\hat{\mathbf{z}}\right) + \beta\left(a_1\hat{\mathbf{x}} + a_1\hat{\mathbf{y}} + a_1\hat{\mathbf{z}}\right) \\
&= \left(\alpha a_1\hat{\mathbf{x}} + \alpha a_1\hat{\mathbf{y}} + \alpha a_1\hat{\mathbf{z}}\right) + \left(\beta a_1\hat{\mathbf{x}} + \beta a_1\hat{\mathbf{y}} + \beta a_1\hat{\mathbf{z}}\right) \\
&= (\alpha a_1 + \beta a_1)\hat{\mathbf{x}} + (\alpha a_1 + \beta a_1)\hat{\mathbf{y}} + (\alpha a_1 + \beta a_1)\hat{\mathbf{z}}
\end{aligned}$$

Because of the distributive property, $(\alpha+\beta)\mathbf{x} = \alpha\mathbf{x} + \beta\mathbf{x}$. Property M4 is satisfied.

Property M5

Multiply 1 and $\mathbf{x}$.

$$\begin{aligned} 1\mathbf{x} &= 1\,(a_1\hat{\mathbf{x}} + a_1\hat{\mathbf{y}} + a_1\hat{\mathbf{z}}) \\ &= a_1\hat{\mathbf{x}} + a_1\hat{\mathbf{y}} + a_1\hat{\mathbf{z}} \\ &= \mathbf{x} \end{aligned}$$

Property M5 is satisfied. All ten properties are satisfied, so the set of vectors with three equal components is a vector space over the complex numbers. The dimension of the vector space is 1 because each of the vectors is spanned by the one basis vector $(1, 1, 1)$.